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Quantisation of velocity-dependent forces $\alpha\dot{x}^2$ and $\alpha\dot{x}^4$

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Abstract. The eigenvalue spectrum $E_l(V, n, \alpha)$ of normalisable states to the equation $(\alpha\hbar/\mathrm{i}m)^n\varphi^{(n-1)}(x)-(\hbar^2/2m)\varphi''(x)+(V(x)-E)\varphi(x)=0$, recently proposed as quantisation of classical forces $-\alpha\dot{x}^n-\partial V/\partial x$, is discussed for even n. For $n=4$ and a rectangular potential well an analytic argument shows that bound states for $0<\alpha<\varepsilon$, $\varepsilon\to 0$, can exist only if $E_l(V, 4, \alpha)\to E_l^{(0)}(V)$ as $\alpha\to 0$ where $E_l^{(0)}(V)$ are the eigenvalues of the corresponding conservative Schrödinger equation ($\alpha=0$). Numerical results indicate $E_l(V, 4, \alpha)-E_l^{(0)}(V)=c_l(V)\alpha^2+\mathrm{O}(\alpha^3)$ as $\alpha\to 0$. For $n=2$, $E_l(V, 2, \alpha)-E_l^{(0)}(V)=\frac{1}{2}\alpha^2\hbar^2m^{-3}$ is exact and independent of $V(x)$, and at a potential step (to the left since $\alpha>0$) a continuous band of normalisable states exists instead of the scattering states for $\alpha=0$.

1. Introduction

Recently we proposed a new quantisation method [1] for the classical equation

$$m\ddot{x}=-\alpha\dot{x}^n-\frac{\partial V}{\partial x}\qquad n>0\text{ integer, }\alpha>0. \tag{1a}$$

We named the method 'semicanonical' since, in contrast with canonical quantisation, the conserved energy

$$E=\tfrac{1}{2}m\dot{x}^2+V(x)+\alpha\int^x\dot{x}^n\,\mathrm{d}x \tag{1b}$$

rather than a (for $\alpha\neq 0$ only mathematical [2] and not unique) Hamilton function is quantised. The further procedure is as in canonical quantisation of the conservative system corresponding to ($1a$) with $\alpha=0$, i.e. the mechanical momentum $m\dot{x}$ (or slightly more generally, the canonical momentum of the corresponding conservative equation) and the energy E are substituted by the differential operators

$$p=m\dot{x}\to-\mathrm{i}\hbar\frac{\partial}{\partial x} \tag{2a}$$

$$E\to\mathrm{i}\hbar\frac{\partial}{\partial t}. \tag{2b}$$

Equations ($1b$) and ($2a$) yield the Hamilton or *energy* operator

$$H=H_e=H_0+\alpha H_i. \tag{3a}$$

$$H_0=\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x) \tag{3b}$$

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is the Hamilton operator of the corresponding conservative equation and

$$\alpha H_i = \alpha \left(\frac{\hbar}{\mathrm{i}m}\right)^n \frac{\partial^{n-1}}{\partial x^{n-1}} \tag{3c}$$

the quantised form of the integral in (1*b*), is a non-Hermitian operator.

In [1] we studied the Schrödinger equation:

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = H_e \psi \tag{4}$$

and saw that eigenstates to H_e decay in time for odd $n = 1$ and 3. However, for $n = 2$ stable eigenstates exist in the oscillator potential $V_{\mathrm{OSC}} = \frac{1}{2}m\omega^2 x^2$ with eigenvalues

$$E_l(V_{\mathrm{OSC}}, 2, \alpha) = \omega\hbar(l + \tfrac{1}{2}) + \frac{\alpha^2\hbar^2}{2m^3} \qquad l = 0, 1, 2, \ldots. \tag{5}$$

The results reflect that the classical force $-\alpha\dot{x}^n$ is dissipative only for odd n when the product $\dot{x}^n\,\mathrm{d}x$ in the integral of (1*b*) is always non-negative ($\dot{x}$ and $\mathrm{d}x = \dot{x}\,\mathrm{d}t$ have the same sign, $\mathrm{d}t > 0$). For even n the product has the same sign as $\dot{x}$, i.e. the integral can add energy to or extract energy from the particle's energy $\frac{1}{2}m\dot{x}^2 + V(x)$, depending on the direction of motion. If the particle during its motion passes several times through some point x then the integral in (1*b*) is, for even n, a unique function of x (which, however, depends on V and the initial values x_0, $\dot{x}_0$) while for odd n it increases monotonically, independent of the direction of motion. In this wider sense, the force $-\alpha\dot{x}^n$, n even, can be considered as conservative. Nevertheless, throughout all the following the term 'conservative' refers only to the Hamilton operator (3*b*), i.e. $H = H_0$.

Concerning the correspondence of our quantum results with the classical ones we would like to add the following to what was said in [1]. For odd n, indeed, the quantum results could seem incomplete as they do not indicate into which states an eigenstate of H_e, excited at $t = 0$, decays. However, even the classical solution to (1*a*)—though it describes the motion of the particle completely—does not provide information about the further destination of the energy lost by the particle. The latter problem can be considered as analogous to the quantum mechanical one of which states an eigenstate of H_e decays into. The answer to both questions would require considering the particles of the dissipative medium, too, and therefore is beyond a one-particle theory, in classical as well as in quantum mechanics.

To satisfy the uncertainty principle and, in the limit $\alpha \to 0$, to agree with the results of the corresponding conservative Schrödinger equation are two essential conditions that the quantum theory based on the Hamilton operator (3*a*) can be considered a consistent theory. The first is guaranteed in the proposed quantisation because of (2), and it is relatively easy to see that the second is true for $n = 1, 2$ and 3 [1]. But for $n \geqslant 4$ the Schrödinger equation with the Hamilton operator (3*a*) is of order higher than two in the spatial variable x. Since the coefficient of the highest-order derivative is proportional to α a singularity of the general solution in $\alpha = 0$ must be expected. This could make impossible a smooth relationship between the results for H_0 and those for H_e in the limit $\alpha \to 0$.

This question will be examined in § 2. Since the difficulties in handling the problem increase rapidly with n we choose $n = 4$ and, to deal with an exactly solvable problem, consider bound states in a rectangular potential well. But bound states for even n are also of proper interest because of the qualitative difference between classical even- and odd-power $\dot{x}^n$ forces. Therefore in § 3 bound states for $n = 2$ will be investigated in a general binding potential and at a potential step. All results can be interpreted in a consistent manner. They will be summarised in § 4.

2. The limit $\alpha \to 0$ and bound states for $n = 4$

For $n = 4$ (3c) and (4) with

$$\psi(x, t) = \varphi(x) \exp(Et/\mathrm{i}\hbar) \tag{6a}$$

yield the stationary Schrödinger equation

$$\varphi'''(x) - \frac{m^3}{2\alpha\hbar^2}\varphi''(x) + \frac{m^4}{\alpha\hbar^4}(V(x) - E)\varphi(x) = 0. \tag{6b}$$

A prime denotes differentiation with respect to x.

2.1. General solution for a rectangular well

Equation (6b) is exactly solvable for a rectangular potential well:

$$V(x) = \begin{cases} 0 & \text{if } x < 0 \\ -V_0 < 0 & \text{if } 0 < x < a \\ 0 & \text{if } a < x. \end{cases} \tag{6c}$$

Three fundamental solutions are $\varphi_{\text{fund},i} = \exp(w_i x)$, $i = 1, 2$ and 3. The constants w_i solve the cubic equation

$$w^3 - \frac{b}{\alpha}w^2 + \frac{k}{\alpha} = 0 \qquad b = \frac{m^3}{2\hbar^2} \qquad k = \frac{m^4}{\hbar^4}(V - E). \tag{7}$$

We are interested in the solutions for energies $-V_0 < E < 0$ and small positive α such that

$$d = 27\alpha^2 k - 4b^3 < 0. \tag{8}$$

For $x < 0$ and $x > a > 0$ the constant k is positive:

$$k = k_1 = -\frac{m^4}{\hbar^4}E > 0. \tag{9a}$$

Equation (8) implies $dk < 0$, and (7) has three real solutions [3]:

$$w_{1,2} = \lambda_{1,2} = \frac{b}{3\alpha}[-2\cos(\gamma \mp \tfrac{1}{3}\pi) + 1] \tag{9b}$$

$$w_3 = \lambda_3 = \frac{b}{3\alpha}(2\cos\gamma + 1) \tag{9c}$$

where

$$\gamma = \tfrac{1}{3}\cos^{-1}(1 - 27\alpha^2 k_1/2b^3). \tag{9d}$$

For $0 < x < a$ the constant k is negative:

$$k = k_2 = -\frac{m^4}{\hbar^4}(V_0 + E) \tag{10a}$$

and $dk > 0$ for all α. In this case two solutions of (7) are conjugate complex and one is real:

$$w_{1,2} = \rho \pm \mathrm{i}\mu \tag{10b}$$

$$w_3 = \nu = \frac{1}{3\alpha}(z^{1/3} + b^2 z^{-1/3} + b). \tag{10c}$$

ρ, μ and z are defined as

$$\rho = \frac{1}{6\alpha}(-z^{1/3} - b^2 z^{-1/3} + 2b) \tag{10d}$$

$$\mu = \frac{\sqrt{3}}{6\alpha}(z^{1/3} - b^2 z^{-1/3}) \tag{10e}$$

$$z = \frac{1}{\sqrt{12}}[9\alpha(27\alpha^2 k_2^2 - 4b^3 k_2)^{1/2} - \sqrt{3}(27\alpha^2 k_2 - 2b^3)] \tag{10f}$$

and $z^{1/3}$ is the real value of the cubic root.

The solutions (9) and (10) behave in the following way as $\alpha \to 0$:

$$\lambda_{1,2} = \mp\left(\frac{k_1}{b}\right)^{1/2} + \mathrm{O}(\alpha) \tag{11a}$$

$$\lambda_3 = \frac{b}{\alpha} + \mathrm{O}(\alpha) \tag{11b}$$

$$\rho = \mathrm{O}(\alpha) \tag{11c}$$

$$\mu = \left(\frac{-k_2}{b}\right)^{1/2} + \mathrm{O}(\alpha) \tag{11d}$$

$$\nu = \frac{b}{\alpha} + \mathrm{O}(\alpha). \tag{11e}$$

Thus, λ_3 and ν are singular in $\alpha = 0$ while $\lambda_{1,2}$ and $\rho \pm \mathrm{i}\mu$ tend to the solutions of the quadratic equation $-\frac{1}{2}\hbar^2 w^2/m + \hbar^4 k_i/m^4 = 0$, $i = 1$ and 2, related to the corresponding conservative Schrödinger equation. From (11) one concludes that, for small positive α, normalisable (square integrable) solutions of (6*b*, *c*) have the form

$$\varphi(x) = \begin{cases} \varphi_1 = A_2 \exp(\lambda_2 x) + A_3 \exp(\lambda_3 x) & \text{if } x < 0 \\ \varphi_2 = (B_1 \sin \mu x + B_2 \cos \mu x)\exp(\rho x) + B_3 \exp(\nu x) & \text{if } 0 < x < a \\ \varphi_3 = C_1 \exp(\lambda_1 x) & \text{if } a < x. \end{cases} \tag{12}$$

2.2. *Eigenvalue equation*

It follows from (6*b*, *c*) that $\alpha\varphi'''$ has a finite discontinuity in $x = 0$ and in $x = a$ where the potential $V(x)$ is not continuous. Then, for $\alpha > 0$, φ'', φ' and φ must be continuous. The six continuity conditions

$$\varphi_1^{(j)}(0) = \varphi_2^{(j)}(0) \qquad \varphi_2^{(j)}(a) = \varphi_3^{(j)}(a) \qquad j = 0, 1, 2 \tag{13}$$

(and the normalisation) determine the six constants A_2, A_3, B_1, B_2, B_3 and C_1 of a bound state (12). Eliminating A_2, A_3, B_1 and C_1, we reduce the system (13) to two homogeneous equations for B_2 and B_3:

$$f_2 B_2 + f_3 B_3 = 0 \tag{14a}$$

$$g_2 B_2 + g_3 B_3 = 0 \tag{14b}$$

where

$$f_2 = (-\mu + s\rho - s\lambda_1)\sin\mu a + (\rho + s\mu - \lambda_1)\cos\mu a \tag{14c}$$

$$f_3 = t(\rho - \lambda_1)\sin\mu a + t\mu\cos\mu a + (\nu - \lambda_1)\exp[(\nu - \rho)a] \tag{14d}$$

$$g_2 = [-2\rho\mu - s(\lambda_1^2 + \mu^2 - \rho^2)]\sin\mu a + (\rho^2 - \mu^2 - \lambda_1^2 + 2s\rho\mu)\cos\mu a \tag{14e}$$

$$g_3 = t(\rho^2 - \mu^2 - \lambda_1^2)\sin\mu a + 2t\rho\mu\cos\mu a + (\nu^2 - \lambda_1^2)\exp[(\nu - \rho)a] \tag{14f}$$

$$s = \frac{\rho^2 - \mu^2 - \lambda_2\rho - \lambda_3\rho + \lambda_2\lambda_3}{\mu(\lambda_2 + \lambda_3 - 2\rho)} \tag{14g}$$

$$t = \frac{\nu^2 - \nu\lambda_2 - \nu\lambda_3 + \lambda_2\lambda_3}{\mu(\lambda_2 + \lambda_3 - 2\rho)}. \tag{14h}$$

The condition that a non-trivial solution $(A_2, A_3, B_1, B_2, B_3, C_1) \neq (0, 0, 0, 0, 0, 0)$ exists is now reduced to the condition that the determinant of (14a, b) vanishes:

$$f_2 g_3 - f_3 g_2 = 0. \tag{15}$$

In § 2.3 it will be examined if and how the solutions $E = E_l(V, 4, \alpha)$ of (15) and the eigenvalues $E_l^{(0)}(V)$, $l = 0, 1, 2, \ldots$, of the conservative Schrödinger equation are related to each other.

2.3. *Solutions of the eigenvalue equation*

From (11) and (14) one sees that s, t, f_2 and g_2 are regular in $\alpha = 0$ while the leading singularities are $\nu^2\exp(\nu a) \sim b^2\alpha^{-2}\exp(ba/\alpha)$ in g_3 and $\nu\exp(\nu a) \sim b\alpha^{-1}\exp(ba/\alpha)$ in f_3. One concludes that, for $0 < \alpha < \varepsilon$, $\varepsilon \to 0$, (15) cannot be satisfied unless $f_2 = O(\alpha)$ as $\alpha \to 0$. Using (11) and (14) one can rewrite this condition in order O(1) as

$$\tan\left[\left(\frac{-k_2}{b}\right)^{1/2} a\right] + 2\frac{(-k_1 k_2)^{1/2}}{k_1 + k_2} = 0. \tag{16}$$

The textbooks of quantum mechanics [4] show that (16) is just the condition which determines the eigenvalues $E_l^{(0)}$ of the bound states to the conservative Schrödinger equation ($\alpha = 0$) with the potential (6c). That is, bound states to (6b, c) cannot exist for arbitrarily small positive values of α unless their eigenvalues $E_l(V, 4, \alpha)$ in the limit $\alpha \to 0$ tend to the eigenvalues $E_l^{(0)}(V)$.

To verify if solutions exist for small positive α we have solved (15) numerically for $V_0 = -1$, $a = 1$ and $V_0 = -10$, $a = 1$, setting $m = \hbar = 1$. In the first example the conservative Schrödinger equation has one bound state for $E = E_0^{(0)}(V) = -0.3079\ldots$, and in the second one two bound states for $E = E_0^{(0)}(V) = -7.7050\ldots$ and $E = E_1^{(0)}(V) = -1.8628\ldots$. Since f_3 and g_3 tend to infinite values as $\alpha \to 0$, the numerical solution has been carried out for $\alpha \geq 0.005$ and in double precision (corresponding to

Table 1. $E_l(V_0)$, $l=0, 1$, are the numerical solutions of (15) for $\alpha > 0$ and of (16) for $\alpha = 0$, all with $a = 1$. The exact solutions $E_l(V, 4, \alpha)$ and $E_l^{(0)}(V)$ lie between $E_l(V_0)$ and $E_l(V_0) - 10^{-8}$. The calculations of $E_0(V_0)$ have not been continued to values of α where $d(k = k_1, \alpha)$ (cf (8)) is positive in the energy region of the bound state. (This would require another ansatz for $x < 0$ and $a < x$ in (12).)

α	$E_0(V_0=-1)$	$E_0(V_0=-10)$	$E_1(V_0=-10)$
0	−0.307 921 59	−7.705 009 25	−1.862 852 23
0.005	−0.307 865 68	−7.703 300 17	−1.853 757 06
0.010	−0.307 701 25	−7.698 391 02	−1.827 468 67
0.020	−0.307 067 71	−7.680 138 56	−1.731 409 99
0.030	−0.306 064 96	−7.652 343 91	−1.593 518 19
0.040	−0.304 739 28	−7.616 828 01	−1.432 651 93
0.049		−7.579 602 73	
0.050	−0.303 137 41	$d(k_1)>0$	−1.263 934 23
0.100	−0.292 460 42		−0.551 375 18
0.150	−0.280 079 56		−0.172 113 42
0.200	−0.267 928 96		−0.025 763 65
0.220	−0.263 300 47		−0.006 030 47
0.240	−0.258 830 52		−0.000 021 08
0.241 34			$-10^{-9} < E_1 < 0$
0.250	−0.256 656 89		
0.270	−0.252 433 22		
0.280	$d(k_1)>0$		

approximately 29 decimal places during computation). The results are shown in table 1 and indicate

$$E_l(V, 4, \alpha) - E_l^{(0)}(V) = c_l(V)\alpha^2 + \mathrm{O}(\alpha^3) \qquad \text{as } \alpha \to +0. \tag{17}$$

The presence of a term proportional to α^3 and higher odd orders of α in (17) does not mean that $E_l(V, 4, \alpha)$ cannot be symmetric with respect to $\alpha = 0$. The results above are, of course, only valid for non-negative α, due to the asymptotic behaviour of (12).

The smoothness (17) of the eigenvalues in $\alpha = 0$ suggests the following behaviour of the coefficients in (12):

$$A_3(\alpha) = \mathrm{O}(\alpha^2) \qquad B_3(\alpha) = \mathrm{O}(\alpha^2 \exp(-ba/\alpha)) \qquad \text{as } \alpha \to +0. \tag{18}$$

Then, in the limit $\alpha \to +0$, $A_3 \exp(\lambda_3 x)$ and $B_3 \exp(\nu x)$ do not contribute to the continuity conditions (13) for $j = 0, 1$ while for $j = 2$ they give a finite non-zero contribution. In other words, the singular part of the solution (12) does not affect the continuity conditions of the conservative problem.

3. Bound states of $H_e = H_0 - \alpha\hbar^2 m^{-2}\partial/\partial x$

3.1. *Bound states in a general binding potential*

For $n = 2$ one obtains from (3) the stationary Schrödinger equation

$$\varphi''(x) + \frac{2\alpha}{m}\varphi'(x) + \frac{2m}{\hbar^2}(E - V(x))\varphi(x) = 0. \tag{19}$$

Let $V(x)$ be a general binding potential and χ_l a bound state of the conservative Schrödinger equation:

$$\chi''(x)+\frac{2m}{\hbar^2}(E-V(x))\chi(x)=0 \tag{20}$$

with eigenvalue $E=E_l^{(0)}(V)$. Setting $\chi_l(x)=\varphi_l(x)\exp(\alpha x/m)$ in (20), one gets

$$\left[\varphi_l''+\frac{2\alpha}{m}\varphi_l'+\left(\frac{2m}{\hbar^2}(E_l^{(0)}-V(x))+\frac{\alpha^2}{m^2}\right)\varphi_l\right]\exp(\alpha x/m)=0 \tag{21a}$$

and concludes that

$$\varphi_l(x)=\chi_l(x)\exp(-\alpha x/m) \tag{21b}$$

is an eigenfunction of (19) with eigenvalue

$$E_l(V,2,\alpha)=E_l^{(0)}(V)+\alpha^2\hbar^2/2m^3. \tag{21c}$$

The result (21b, c) generalises (5), found for the harmonic oscillator [1] and is not restricted to bound states χ_l and/or φ_l only.

In order that, with χ_l, φ_l also be a bound state two conditions must be fulfilled:

$$E_l^{(0)}+\frac{\alpha^2\hbar^2}{2m^3}<V(\pm\infty) \tag{22a}$$

$$\int_{-\infty}^{x}|\varphi_l|^2\,\mathrm{d}x<\infty \qquad -\infty<x. \tag{22b}$$

They are satisfied for the harmonic oscillator since $V(\pm\infty)=\infty$ and $\chi_l\sim x^l\exp(-\frac{1}{2}m\omega x^2\hbar^{-1})$ as $x\to\pm\infty$. For a potential which has a constant value $V(x)=V(\pm\infty)$ for large arguments $|x|>x_0$, both conditions are equivalent. This can be seen from the asymptotic behaviour

$$\chi_l\sim\exp\{[-2m(E_l^{(0)}-V(\pm\infty))]^{1/2}x/\hbar\} \qquad \text{if } x<-x_0. \tag{23}$$

3.2. Bound states at a potential step

Let us now consider (19) for a potential step:

$$V(x)=\begin{cases}0 & \text{if } x<0\\ -V_0<0 & \text{if } x>0.\end{cases} \tag{24}$$

In contrast to the conservative case $\alpha=0$, for $\alpha>0$ square integrable solutions with energies $-V_0<E<0$ exist to (19) and (24):

$$\varphi(x)=\begin{cases}\varphi_-=A\exp[(\gamma-\alpha/m)x] & \text{if } x<0\\ \varphi_+=\begin{cases}D_1\exp[(\sqrt{\delta}-\alpha/m)x]+D_2\exp[(-\sqrt{\delta}-\alpha/m)x] & \text{if } \delta>0, x>0\\ (D_3+D_4x)\exp(-\alpha x/m) & \text{if } \delta=0, x>0\\ (D_5\sin\sqrt{-\delta}x+D_6\cos\sqrt{-\delta}x)\exp(-\alpha x/m) & \text{if } \delta<0, x>0\end{cases}\end{cases} \tag{25a}$$

where

$$\gamma=\left(\frac{\alpha^2}{m^2}-\frac{2mE}{\hbar^2}\right)^{1/2}>\frac{\alpha}{m} \tag{25b}$$

$$\delta=\frac{\alpha^2}{m^2}-\frac{2m}{\hbar^2}(E+V_0)<\frac{\alpha^2}{m^2}. \tag{25c}$$

For all energies $-V_0 < E < 0$, φ_- and φ_+ can be joined in $x=0$ such that φ and φ' are continuous and φ is normalised to 1. The binding mechanism at large positive x works for all positive α. With decreasing values of α the binding strength becomes weaker, and the probability to find the particle at large positive values of x increases. Clearly, this binding mechanism would work at a potential step on the RHS, i.e. $V(x) = -V_0 < 0$ if $x<0$ and $V(x)=0$ if $x>0$, only if $\alpha<0$.

We emphasise that the bound states have a continuous band $-V_0 < E < 0$ of eigenvalues since both fundamental solutions of (19) and (24) decay exponentially as $x \to +\infty$ (cf (25c)). For $\delta \geqslant 0$ they have energies $-V_0 < E \leqslant -V_0 + \frac{1}{2}\alpha^2\hbar^2 m^{-3}$ and arise from physically uninteresting solutions of (20) with energy $E^{(0)} \leqslant -V_0$. Bound states (25) with $\delta<0$ correspond to scattering states of (20) with $-V_0 < E^{(0)} < -\frac{1}{2}\hbar^2\alpha^2 m^{-3}$ (in agreement with (21)). In the limit $\alpha \to 0$ the bound states (25) tend to the scattering states of (20) since then only $\delta < 0$ is realised and $\exp(-\alpha x/m) \to 1$.

A continuous band $-V_0 < E < 0$ of bound states at the potential step (24) is also expected for $n=4$ in (3c) ($\alpha>0$): namely, if $\lambda_{1,2,3}$ are the solutions of (7) for $x<0$, it follows from $-(\lambda_1+\lambda_2+\lambda_3) = -b/\alpha < 0$ and $-\lambda_1\lambda_2\lambda_3 = k/\alpha > 0$ that two of them are positive or have a positive real part. For $x>0$ one can see from (10a, f) that $z>b$ and then from (10d) and the inequality $y+1/y>2$, valid for positive $y \neq 1$, that $\rho<0$, i.e. two solutions of (7) have a negative real part (while the third solution is positive). Thus there are four constants available to make φ, φ' and φ'' continuous in $x=0$ (and to normalise φ). On the other hand, for $n=1$ one will find scattering states at the potential step which decay in time as $\exp(-\alpha t/m)$ [1] since the classical force $-\alpha\dot{x}$ is dissipative for $\alpha>0$.

For comparison let us consider a classical particle obeying (1a) with (24) and even n. Suppose that its initial position $x_0 > 0$ and velocity $\dot{x}_0 \neq 0$ are such that the motion is restricted by (24) to the positive half-axis $x>0$. Then, for $t \to \infty$ the particle tends to $x=+\infty$ with velocity $\dot{x} \to 0$. The motion is not bounded to a finite region, and the particle loses energy. To avoid both, a small perturbation would be necessary which could give the particle a small velocity in the negative direction. In quantum mechanics such small perturbations are always present in the form of small velocity fluctuations, as a consequence of the uncertainty principle. For this reason stable bound states of the energy operator (3a–c) can exist at the potential step (24) for $n=2$ and are expected to exist also for greater even n.

4. Summary

We have discussed the quantisation of velocity-dependent forces $\alpha\dot{x}^2$ and $\alpha\dot{x}^4$. One main result is that also for (integer) $n>3$ agreement with the conservative quantum results can be obtained in the limit $\alpha \to 0$, as shown explicitly for a rectangular potential well and $n=4$. This result also confirms the existence of stable bound states for an even n different from $n=2$. As for $n=2$ also for $n=4$ the energy levels of H_e are higher than the corresponding levels of H_0, due to the exchange energy between the particle and the medium which provides the interaction αH_i. But for $n=4$ the energy shift depends on the potential $V(x)$ and on the quantum number of the bound state.

For $n=2$ the quantitative relationship (21b, c) between the solutions of the conservative Schrödinger equation (20) and those of (19) has been shown for general potentials $V(x)$. A new effect was found at a potential step (to the left). There, instead of the scattering states for $\alpha = 0$, a continuous band of bound states exists in the presence of the interaction $-\alpha\hbar^2 m^{-2}\partial/\partial x$. Such a binding effect is also predicted for $n = 4$.

The existence of stable bound states is readily interpretable by the fact that, for even n, the classical force $-\alpha\dot{x}^n$ is not dissipative and by the uncertainty principle. Since for odd $n = 1, 3$ one-particle quantum states decay in time [1] we conclude that the proposed quantisation of (1*b*) reflects the qualitative difference between even- and odd-power $\dot{x}^n$ forces present on the classical level. In this sense the results presented here for even n are complementary to the former ones for odd n and can serve as one more (this time *a posteriori*) justification of the method.

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References

[1] Geicke J 1989 *J. Phys. A: Math. Gen.* **22** 1017
[2] Ray J R 1979 *Lett. Nuovo Cimento* **25** 47, **26** 64
[3] Korn G A and Korn T M 1968 *Mathematical Handbook for Scientists and Engineers* (New York: McGraw Hill) pp 22-3
[4] Macke W 1962 *Quanten* (Leipzig: Akademische Verlagsgesellschaft) pp 33-4